A New Technique for Deriving Electric Fields from Sequences of Vector Magnetograms

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Electric fields on the solar surface determine the flux of magnetic energy and relative magnetic helicity into flare and CME-producing parts of the solar atmosphere:

$$\frac{\partial E_M}{\partial t} = \iint_{\mathbf{S}} dS\, \hat{\mathbf{n}} \cdot \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \cong \iint_{\mathbf{S}} dS\, \hat{\mathbf{n}} \cdot \frac{-1}{4\pi} (\mathbf{v} \times \mathbf{B}) \times \mathbf{B},$$

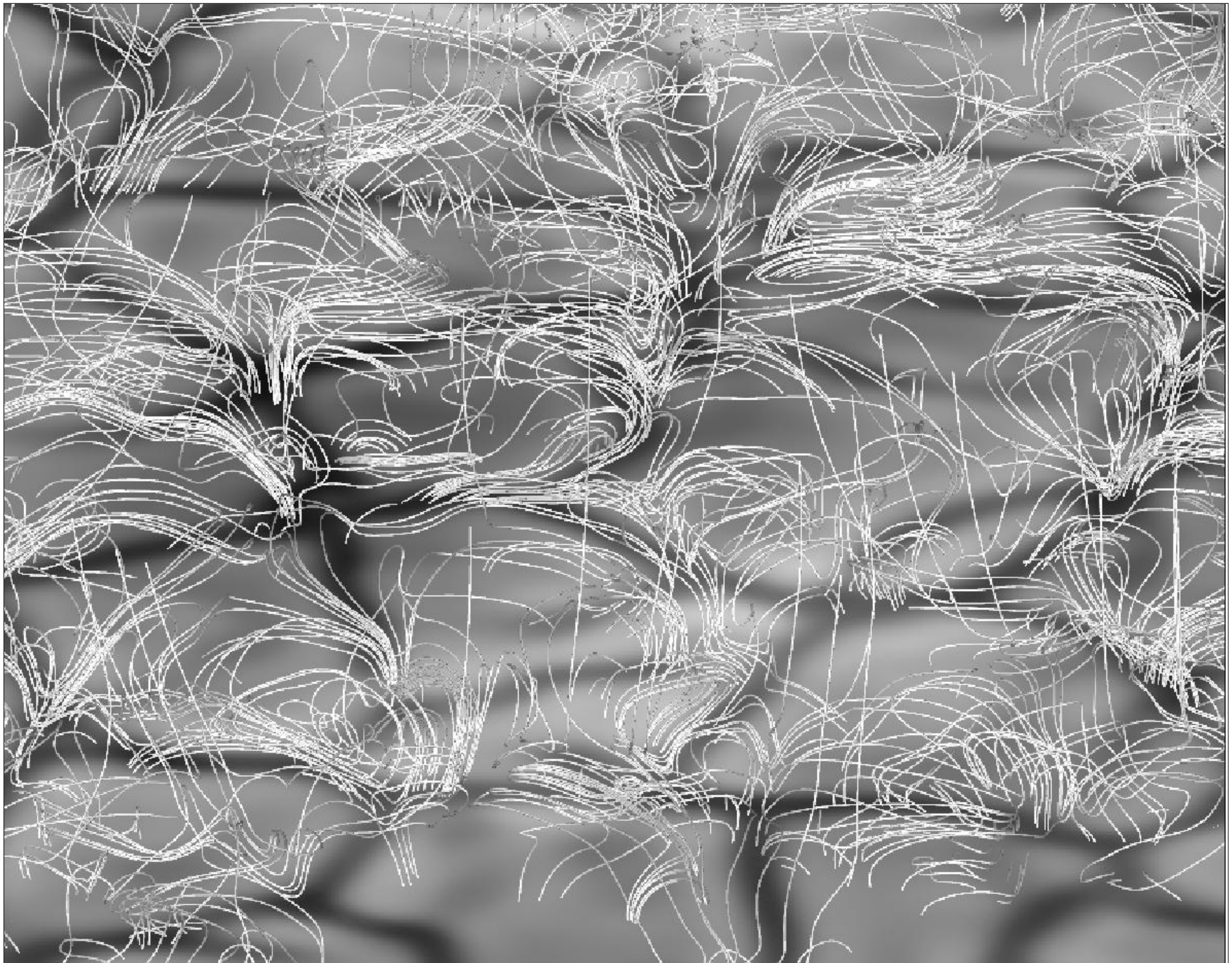
$$\frac{\partial E_F}{\partial t} \cong \frac{1}{4\pi} \iint_{\mathbf{S}} dS\, B_n \mathbf{u}_h \cdot (\mathbf{B}_h^{(P)} - \mathbf{B}_h), \text{ where } B_n \mathbf{u}_h \equiv (\mathbf{v}_h B_n - \mathrm{v}_n \mathbf{B}_h),$$

$$\frac{dH}{dt} = 2 \iint_{\mathbf{S}} dS\, \hat{\mathbf{n}} \cdot \mathbf{A_p} \times \mathbf{E} \cong 2 \iint_{\mathbf{S}} dS \left\{ (\mathbf{A_P} \cdot \mathbf{B_h}) \mathrm{v}_n - (\mathbf{A_P} \cdot \mathbf{v_h}) B_n \right\}.$$

Here, $\partial E_M/\partial t$ is the change in magnetic energy in the solar atmosphere, $\partial E_F/\partial t$ is the difference between the rate of change of total magnetic energy and the potential-field magnetic energy, given a surface distribution of $\mathbf{U}_h$ (Welsch 2006, ApJ 638, 1101), and dH/dt is the change of magnetic helicity of the solar atmosphere.

The flow field **v** is important because to a good approximation, **E** =-**v**/c x **B** in the layers where the magnetic field is measured. Here **E** is the electric field, **B** is the magnetic field, and $\mathbf{A_p}$ is the vector potential of the potential magnetic field that matches its measured normal component.

Flow fields and electric fields provide needed physical boundary conditions for data-driven or assimilative MHD models of the solar atmosphere



Approaches to Computing Electric Fields from Magnetograms

- Assume **E**=-**v**/cx**B** and find v from local correlation tracking techniques applied to changes in line-of-sight magnetograms (e.g. FLCT method of Fisher & Welsch)
- Use vector magnetograms and normal component of induction equation to determine 3 components of **v** (e.g. ILCT method of Welsch et al (2004)and MEF method of Longcope (2004)
- Use vector magnetograms to solve all 3 components of the induction equation (main topic of this talk)

How much information about the magnetic induction equation can one extract from a time sequence of (error-free) vector magnetograms taken in a single layer?

$$\frac{\partial B_x}{\partial t} = c\left(\boxed{\frac{\partial E_y}{\partial z}} - \frac{\partial E_z}{\partial y}\right)$$

$$\frac{\partial B_y}{\partial t} = c\left(\frac{\partial E_z}{\partial x} - \boxed{\frac{\partial E_x}{\partial z}}\right)$$

$$\frac{\partial B_z}{\partial t} = c\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

Kusano et al. (2002, ApJ 577, 501) stated that only the equation for the normal component of B (B_z) can be constrained by sequences of vector magnetograms, because measurements in a single layer contain no information about vertical derivatives. Nearly all current work on deriving flow fields or electric fields make this same assumption. **But is this statement true?**

Can we use the other components of the magnetic induction equation?

To investigate this question, we have found it is useful to use the "poloidal-toroidal" decomposition (henceforth PTD) of the magnetic field and its time derivatives. This formalism has been used extensively in the dynamo community, and in anelastic 3D MHD codes such as ASH and ANMHD to ensure that the magnetic field is solenoidal. Before considering time variability and the induction equation, we first use this formalism to describe the magnetic field: $\mathbf{B} = \nabla \times \nabla \times \beta \hat{\mathbf{z}} + \nabla \times \mathcal{J} \hat{\mathbf{z}} \quad (1)$

Here, β is the "poloidal" potential, and $\mathcal{J}$ is the "toroidal" potential. We now show how, starting from a single vector magnetogram, one can derive the potential functions β, ∂β/∂z, and $\mathcal{J}$. Here, we use Cartesian coordinates, but this approach can be done in spherical coordinates as well.

β and $\mathcal{J}$ have the nice properties that their horizontal laplacians can be directly related to the vertical magnetic field component and the vertical current density:

$$\nabla_h^2 \beta = -\mathrm{B}_z \quad (2);$$

$$\nabla_h^2 \mathcal{J} = -\frac{4\pi J_z}{c} = -\hat{\mathbf{z}} \cdot (\nabla_h \times \mathbf{B}_h) \quad (3)$$

Now, apply the horizontal divergence operator ∇_h to the magnetic field **B**:

$$\nabla_h \cdot \mathbf{B} = \nabla_h \cdot \mathbf{B}_h = \nabla_h^2 \left(\frac{\partial \beta}{\partial z} \right) \quad (4)$$

The Poisson equation (4) can be understood physically through the solenoidal constraint on B: The left hand side of equation (4) must be equal and opposite to the vertical derivative of B_z. Since the horizontal laplacian of β is $-B_z$ it follows that the horizontal laplacian of $\partial\beta/\partial z$ must be $-\partial B_z/dz$.

From a 2-D map of the 3 components of **B**, one can solve the three Poisson equations (2-4) for β, $\partial\beta/\partial z$, and $\mathcal{J}$, subject to appropriate boundary conditions. Note that some information about vertical derivatives in the solution was obtained from equation (4).

What are the correct boundary conditions for $\mathcal{J}$, ∂β/∂z, and β?

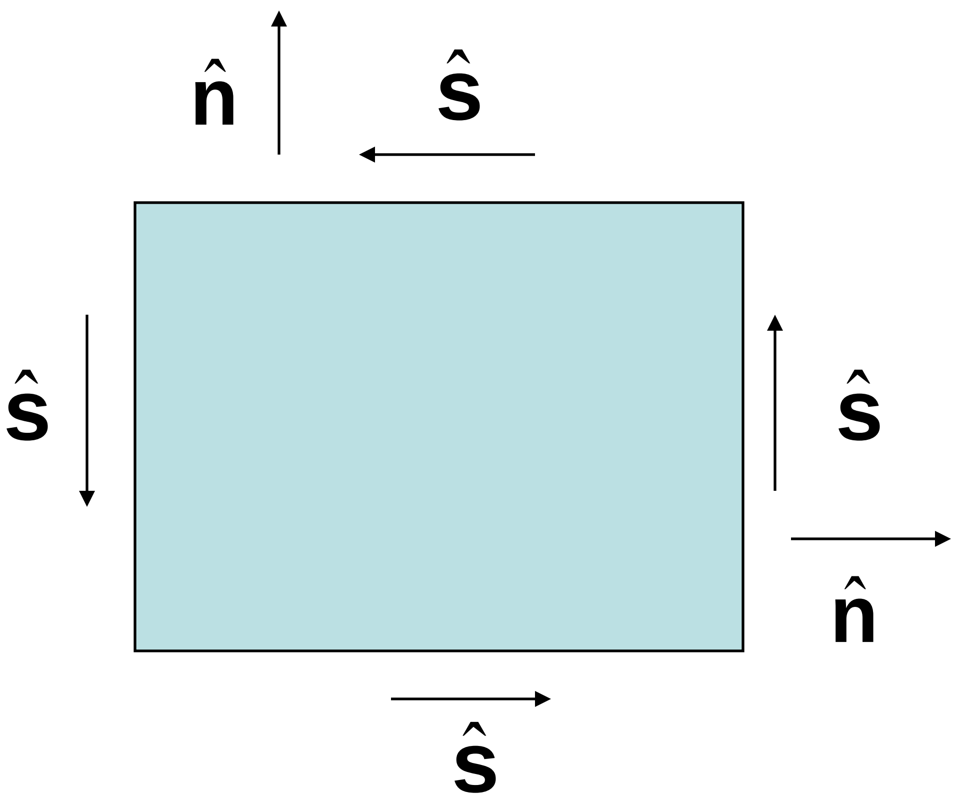
The transverse components of the magnetic field are determined entirely by $\mathcal{J}$ and ∂β/∂z. Equation (1) can be re-written

$$\mathbf{B} = \nabla_h \left(\frac{\partial \beta}{\partial z} \right) + \nabla_h \times \mathcal{J} \hat{\mathbf{z}} - \nabla_h^2 \beta \hat{\mathbf{z}}, \quad (5)$$

from which we derive these coupled von-Neumann boundary conditions:

$$\frac{\partial}{\partial n}\left(\frac{\partial \beta}{\partial z} \right) = B_n - \frac{\partial \mathrm{J}}{\partial s} \qquad (6)$$

$$\frac{\partial \mathrm{J}}{\partial n} = -B_s + \frac{\partial}{\partial s}\left(\frac{\partial \beta}{\partial z} \right) \qquad (7)$$



Here, ∂/∂n denotes derivatives normal to the magnetogram boundary, and ∂/∂s denotes derivatives along the boundary.

The Boundary conditions for β itself do not affect the derived values of the magnetic field B, since it only affects the field component B_z through the Poisson equation (2) itself. But boundary conditions for β do affect the solution for the vector potential:

$$\mathbf{A} = \nabla \times \beta \hat{\mathbf{z}} + \mathcal{J} \hat{\mathbf{z}} - \nabla \Lambda. \quad (8)$$

Here, Λ is a scalar (gauge) potential, left unspecified.

To Summarize: We have shown exactly how one can take knowledge of the vector magnetic field within a bounded 2-d region, and solve 3 Poisson equations, using the boundary conditions that match the observed magnetic field along the boundaries of the magnetic field map. One can in fact reproduce the input magnetic field after the fact from the solutions of the 3 Poisson equations.

Now, consider what happens when we replace **B** by its time derivative, $\partial \mathbf{B}/\partial t$, in equation (1): All of the formalism we have just done will carry through in exactly the same way – we will derive three Poisson equations, analagous to equations (2-4). The difference is that the solutions to these Poisson equations contain information about all 3 components of the magnetic induction equation.

On to the induction equation…

Performing the substitution just described, we derive these Poisson equations relating the time derivative of the observed magnetic field **B** to corresponding time derivatives of the potential functions:

$$\nabla_h^2 \dot{\beta} = -\dot{\mathrm{B}}_z \quad (9);$$

$$\nabla_h^2 \dot{\mathcal{J}} = -\frac{4\pi \dot{J}_z}{c} = -\hat{\mathbf{z}} \cdot (\nabla_h \times \dot{\mathbf{B}}_h) \quad (10);$$

$$\nabla_h^2 \left(\frac{\partial \dot{\beta}}{\partial z} \right) = \nabla_h \cdot \dot{\mathbf{B}}_h \quad (11)$$

The boundary conditions for equations (10) and (11) are specified by the time derivatives of the horizontal fields at the boundaries. The boundary condition for equation (9) is not constrained by the observed time derivatives of the magnetic fields. *These equations and boundary conditions parallel exactly the case for the potentials that describe the magnetic field itself in equations (2-4).*

Since the time derivative of the magnetic field is equal to -c∇xE, we can immediately relate the curl of E and E itself to the potential functions determined from the 3 Poisson equations:

Relating ∇xE and E to the 3 potential functions:

$$\nabla \times \mathbf{E} = \frac{-1}{c}\nabla_h\left(\frac{\partial \dot{\beta}}{\partial z}\right) - \frac{1}{c}\nabla_h \times \dot{\mathcal{J}}\,\hat{\mathbf{z}} + \frac{1}{c}\nabla_h^2 \dot{\beta}\,\hat{\mathbf{z}} \quad (12)$$

$$\mathbf{E} = \frac{-1}{c}\left(\nabla_h \times \dot{\beta}\,\hat{\mathbf{z}} + \dot{\mathcal{J}}\,\hat{\mathbf{z}}\right) - \nabla\psi = \mathbf{E}' - \nabla\psi \quad (13)$$

The expression for **E** in equation (13) is obtained simply by uncurling equation (12). Note the appearance of the 3-d gradient of an unspecified scalar potential ψ.

The induction equation can be written in component form to illustrate precisely where the depth derivative terms $\partial E_y/\partial z$ and $\partial E_x/\partial z$ occur:

$$\frac{\partial B_x}{\partial t} = c\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right) = \frac{\partial}{\partial x}\frac{\partial \dot{\beta}}{\partial z} + \frac{\partial \dot{\mathcal{J}}}{\partial y} \quad (14)$$

$$\frac{\partial B_y}{\partial t} = c\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) = \frac{\partial}{\partial y}\frac{\partial \dot{\beta}}{\partial z} - \frac{\partial \dot{\mathcal{J}}}{\partial x} \quad (15)$$

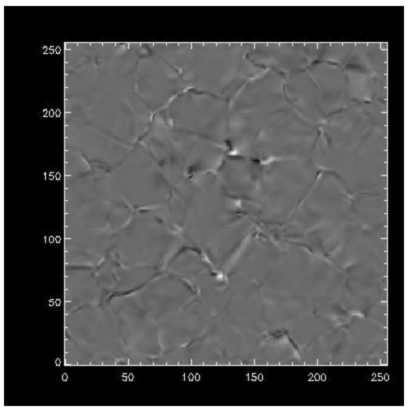
$$\frac{\partial B_z}{\partial t} = c\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) = -\nabla_h^2 \dot{\beta}. \quad (16)$$

Note that these terms originate from the horizontal divergence of time derivatives of the horizontal field (see the discussion following equation 4).

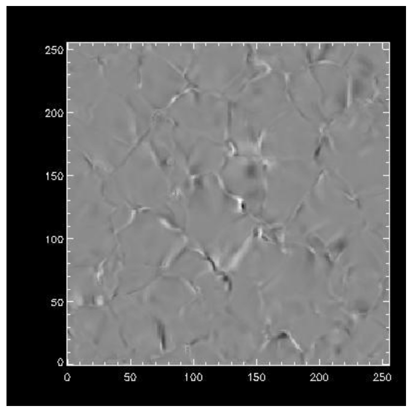
Does it work?

First test: From $\partial B_x/\partial t$, $\partial B_y/\partial t$, $\partial B_z/\partial t$ computed from Bill's RADMHD simulation of the Quiet Sun, solve the 3 Poisson equations with boundary conditions as described, and then go back and calculate $\partial \mathbf{B}/\partial t$ from equation (12) and see how well they agree. Solution uses Newton-Krylov technique:

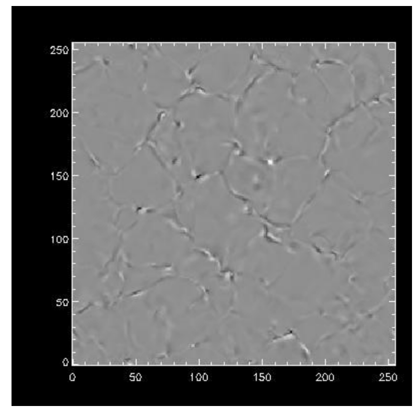
$\partial B_x/\partial t$ RADMHD



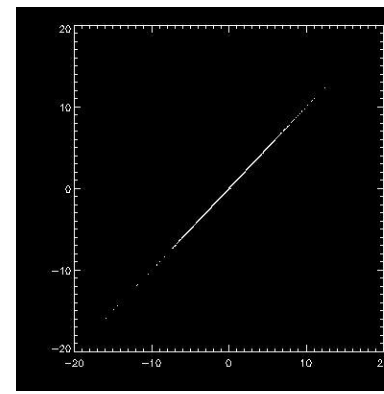
$\partial B_y/\partial t$ RADMHD



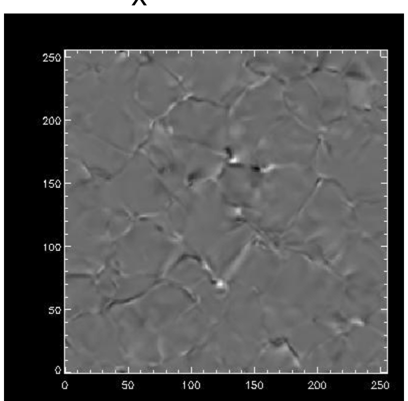
$\partial B_z/\partial t$ RADMHD



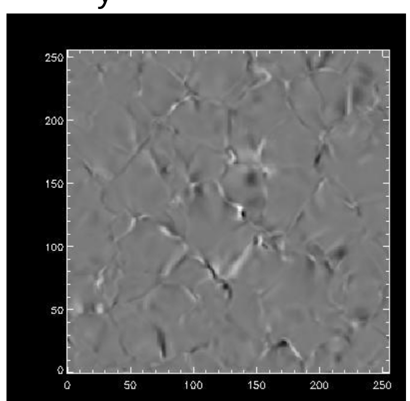
$\partial B_z/\partial t$ vs $\partial B_z/\partial t$



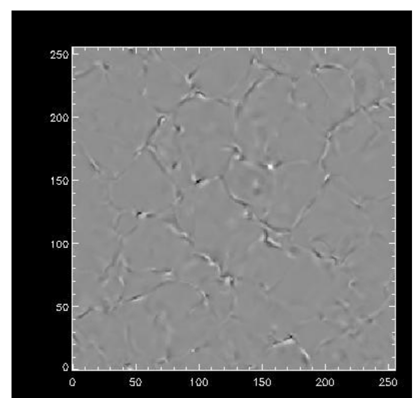
$\partial B_x/\partial t$ derived



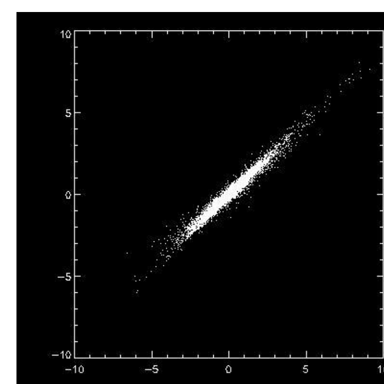
$\partial B_y/\partial t$ derived



$\partial B_z/\partial t$ derived

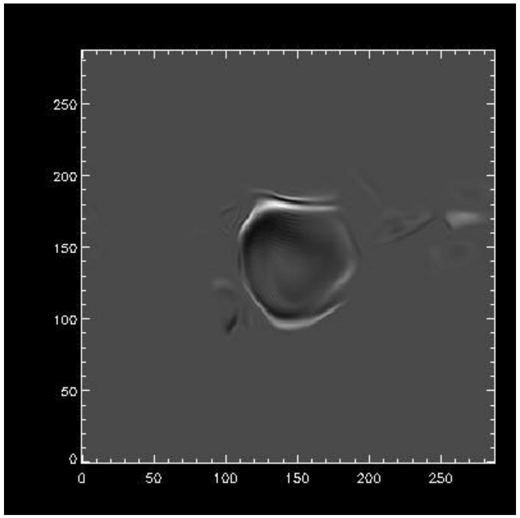


$\partial B_x/\partial t$ vs $\partial B_x/\partial t$

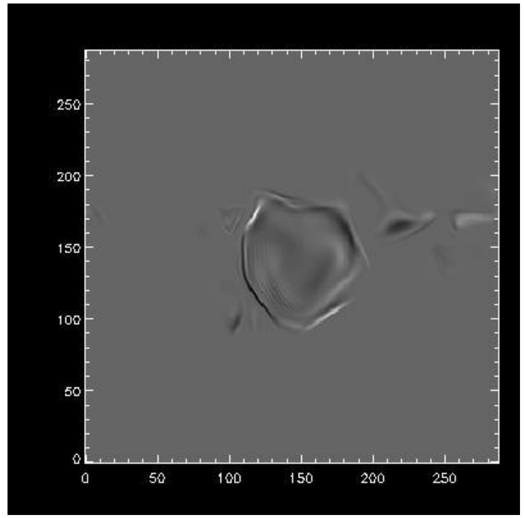


Comparison to velocity shootout case:

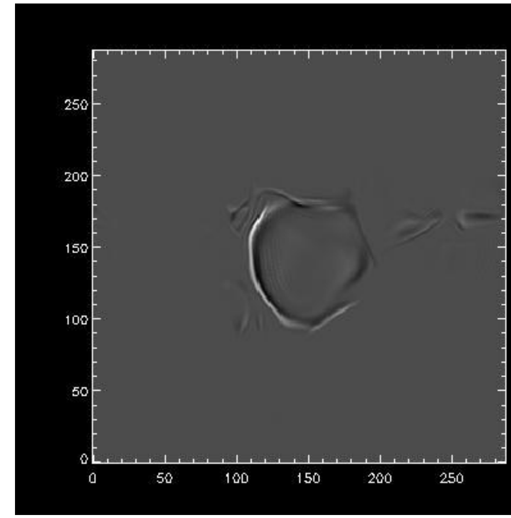
$\partial B_x/\partial t$ ANMHD



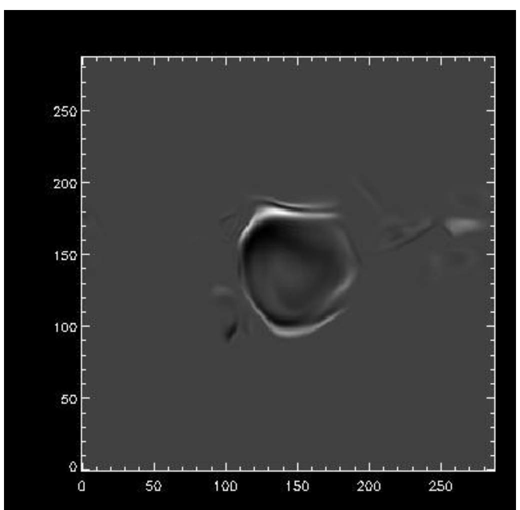
$\partial B_y/\partial t$ ANMHD



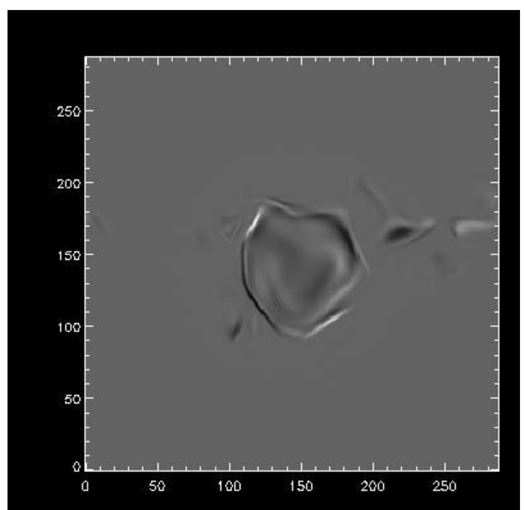
$\partial B_z/\partial t$ ANMHD



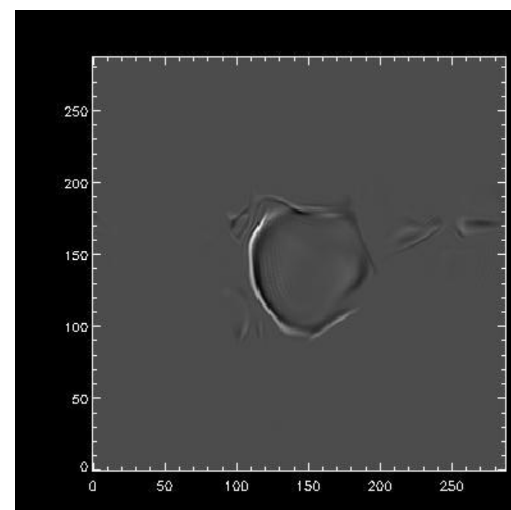
$\partial B_x/\partial t$ derived



$\partial B_y/\partial t$ derived

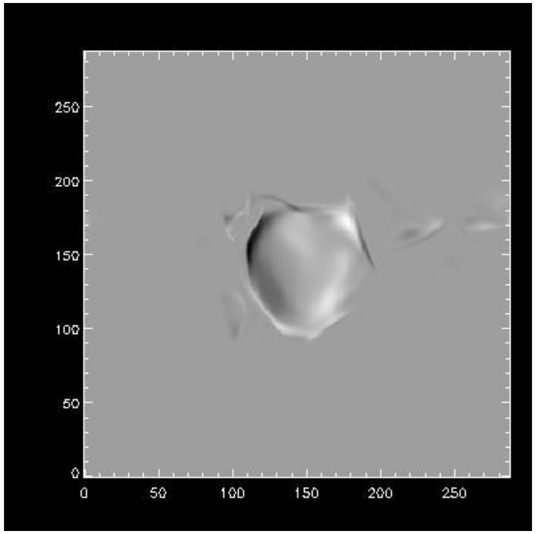


$\partial B_z/\partial t$ derived

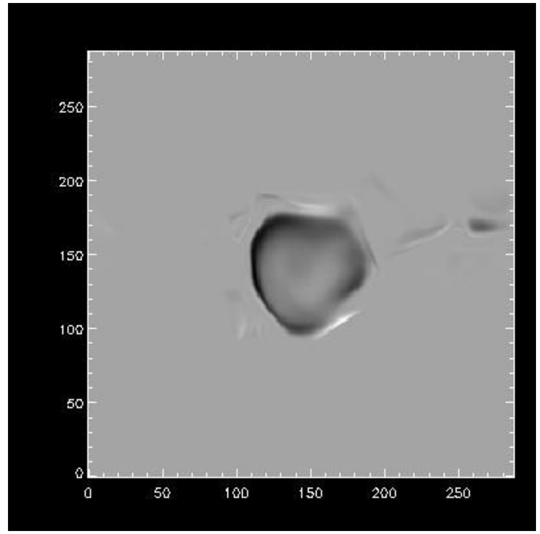


Velocity shoot out case (cont'd)

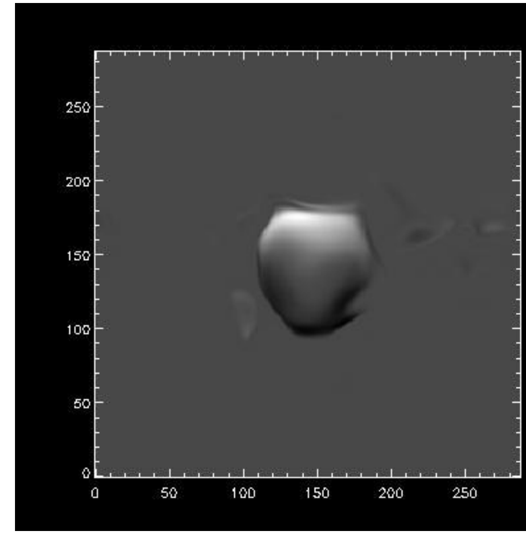
E_x



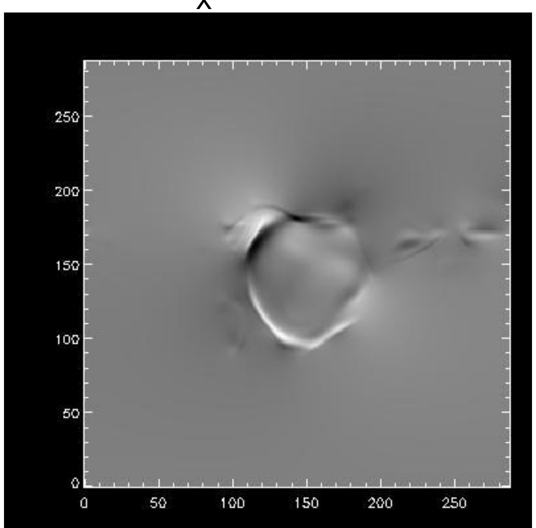
E_y



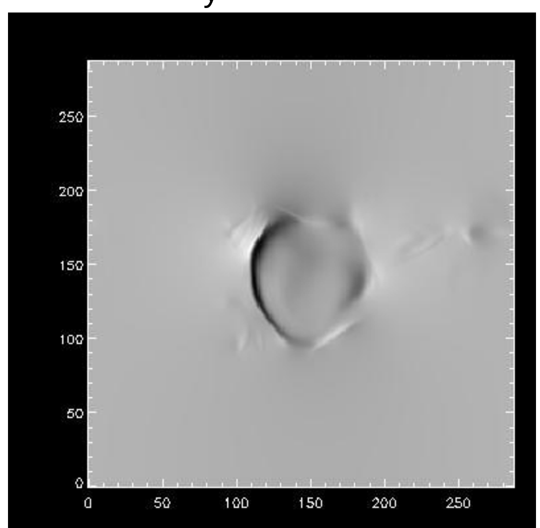
E_z



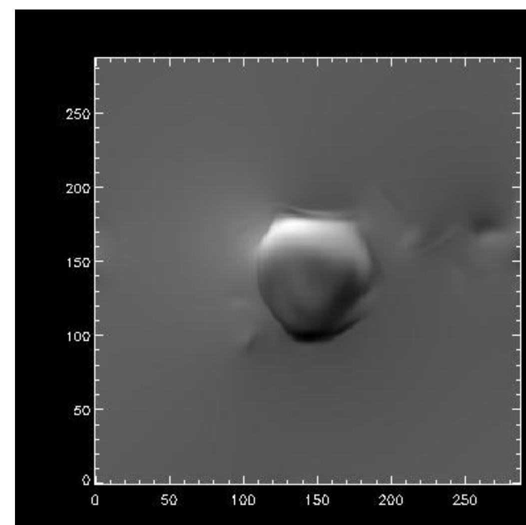
E_x derived



E_y derived



E_z derived



Summary: excellent recovery of ∇xE, only approximate recovery of E.

Why is this? The problem is that **E**, in contrast to ∇**xE**, is mathematically under-constrained. The gradient of the unknown scalar potential in equation (13) does not contribute to ∇**xE**, but it does contribute to **E**.

In the two specific cases just shown, the actual electric field originates largely from the ideal MHD electric field –**v**/c x **B**. In this case, **E·B** is zero, but the recovered electric field contains significant components of **E** parallel to **B**. The problem is that the physics necessary to uniquely derive the input electric field is missing from the PTD formalism. To get a more accurate recovery of **E**, we need some way to add some knowledge of additional physics into a specification of $\nabla\psi$.

We will now show how simple physical considerations can be used to derive constraint equations for ψ.

One Approach to finding ψ: A Variational Technique

The electric field, or the velocity field, is strongly affected by forces acting on the solar atmosphere, as well as by the strong sources and sinks of energy near the photosphere. Here, with only vector magnetograms, we have none of this detailed information available to help us resolve the degeneracy in E from $\nabla\psi$.

One possible approach is to vary ψ such that an approximate Lagrangian for the solar plasma is minimized. The Lagrangian for the electromagnetic field itself is E^2-B^2, for example. The contribution of the kinetic energy to the Lagrangian is ½ ρv^2, which under the assumption that $\mathbf{E} = -\mathbf{v}/c \times \mathbf{B}$, means minimizing E^2/B^2. Since B is already determined from the data, minimizing the Lagrangian essentially means varying ψ such that E^2 or E^2/B^2 is minimized. Here, we will allow for a more general case by minimizing W^2E^2 integrated over the magnetogram, where W^2 is an arbitrary weighting function.

A variational approach (cont'd)

$$\min \iint dx\,dy\,W^2\left((E_x^I - \frac{\partial \psi}{\partial x})^2 + (E_y^I - \frac{\partial \psi}{\partial y})^2 + (E_z^I - \frac{\partial \psi}{\partial z})^2\right)$$

$$\equiv \min \iint dx\,dy\,L(x,y) \quad (22)$$

Here the x,y,z components of E^I are assumed to be taken from equation (13) without the $\nabla\psi$ contribution.

To determine $\partial\psi/\partial z$ contribution to the Lagrangian functional, we can use the relationship **E·B**=**R·B** where **R** is any non-ideal contribution to **E**. Performing the Euler-Lagrange minimization of equation (22) results in this equation:

$$\frac{\partial}{\partial x}\frac{\partial L}{\partial\left(\frac{\partial \psi}{\partial x}\right)} + \frac{\partial}{\partial y}\frac{\partial L}{\partial\left(\frac{\partial \psi}{\partial y}\right)} = 0 \quad (23)$$

A variational approach (cont'd)

Evaluating the equation explicitly results in this elliptic 2nd order differential equation for ψ:

$$\nabla_h \cdot W^2 \Big((\mathbf{E}_h^I - \nabla_h \psi) + \mathbf{B}_h (\mathbf{B}_h \cdot (\mathbf{E}_h^I - \nabla_h \psi) - \mathbf{R} \cdot \mathbf{B}) / B_z^2 \Big) = 0 \qquad (24)$$

We have been pursuing numerical solutions of this equation, along with the 3 Poisson equations described earlier. Comparisons with the original ANMHD electric fields have been poor thus far. This may be due to numerical problems associated with the large dynamic range of the magnetic field-dependent coefficients in this equation.

A possibly more promising approach was recently suggested by Brian Welsch. Writing $E_h = E_h^I - \nabla_h \psi$, and noting that $E_z B_z = \mathbf{R \cdot B - B_h \cdot E_h}$, equation (24) can be re-written and simplified as

$$\nabla_h \cdot \Big((W^2 / B_z)(\boldsymbol{E} \times \boldsymbol{B}) \times \hat{\boldsymbol{z}} \Big) = 0 \qquad (25)$$

or

$$\hat{\boldsymbol{z}} \cdot \Big(\nabla_h \times ((W^2 / B_z) \boldsymbol{E} \times \boldsymbol{B}) \Big) = 0. \qquad (26)$$

A variational approach (cont'd)

Equation (26) implies that we can write

$$(W^2/B_z)(c\mathbf{E}\times\mathbf{B})_h = (W^2/B_z)(c\mathbf{E_h}\times B_z\hat{\mathbf{z}} + cE_z\hat{\mathbf{z}}\times\mathbf{B_h}) = -\nabla_h\chi \quad (27)$$

Dividing by W^2 and then taking the divergence of this equation, we derive an equation for χ that involves the magnetic field or its time derivatives:

$$\frac{\partial B_z}{\partial t} + \nabla_h\cdot\left(\frac{c\mathbf{R}\cdot\mathbf{B}}{B^2}\hat{\mathbf{z}}\times\mathbf{B}_h\right) +$$

$$\nabla_h\cdot\left(\frac{(\mathbf{B_h}\cdot\nabla_\mathbf{h}\chi)}{W^2B^2}\mathbf{B}_h\right) = -\nabla_h\cdot\left(\frac{B_z^2\nabla_h\chi}{W^2B^2}\right) \quad (28)$$

A variational approach (cont'd)

If the non-ideal part of the electric field R is known or if one desires to specify how it varies over the magnetogram field of view, equation (28) can incorporate non-ideal terms in its solution. However, we anticipate that most of the time, the non-ideal term will be much smaller than ideal contributions, and can be set to 0.

Once χ has been determined, it is straightforward to derive the electric field and the Poynting flux **S**=(c/4π)**E**x**B**:

$$\frac{cE_z}{B_z} = \frac{c\mathbf{R}\cdot\mathbf{B}}{B^2} - \frac{\nabla_h \chi \cdot \mathbf{z}\times\mathbf{B_h}}{W^2 B^2} \quad (29)$$

$$c\mathbf{E_h} = \frac{c\mathbf{R}\cdot\mathbf{B}}{B^2}\mathbf{B}_h - \frac{B_z^2}{W^2 B^2}\hat{\mathbf{z}}\times\nabla_h \chi - \frac{\mathbf{B_h}\cdot\nabla_{\mathbf{h}}\chi}{W^2 B^2}\hat{\mathbf{z}}\times\mathbf{B}_h \quad (30)$$

$$\mathbf{S}_h = -\frac{B_z}{4\pi W^2}\nabla_h \chi \quad (31); \quad S_z = \frac{\nabla_h \chi \cdot \mathbf{B_h}}{4\pi W^2} \quad (32)$$

A variational approach (cont'd)

Finally, if one can ignore the non-ideal $\mathbf{R}\times\mathbf{B}$ term in equation (28), then the two special cases of W^2B^2=1(minimized kinetic energy) and W^2=1 (minimized electric field energy) result in these simplified versions of equation (28) for χ:

$$(W^2B^2=1:)\quad \nabla_h\cdot\left(B_z^2\nabla_h\chi\right)+\nabla_h\cdot\left(\left(\mathbf{B_h}\cdot\nabla_h\chi\right)\mathbf{B_h}\right)=\frac{\partial B_z}{\partial t}\qquad(33)$$

$$(W^2=1:)\quad \nabla_h\cdot\left(b_z^2\nabla_h\chi\right)+\nabla_h\cdot\left(\left(\mathbf{b_h}\cdot\nabla_h\chi\right)\mathbf{b_h}\right)=\frac{\partial B_z}{\partial t}\qquad(34)$$

Boundary conditions for χ are not clear, but if the outer boundary of the magnetogram has small or zero magnetic field values, it is likely that the horizontal Poynting flux, and hence $\nabla_h\chi$, has a small flux normal to the magnetogram boundary. Therefore we anticipate that Neumann boundary conditions are the most appropriate for applying to χ if the boundaries are in low field-strength regions.

A variational approach (cont'd)

Finally, how do we relate the variational solution for the total field $E=E^I - \nabla_h\psi$ back to the PTD solutions for $\partial\beta/\partial t$ and $\partial J/\partial t$ and the potential function ψ? The two solutions differ by $\nabla\psi$, allowing us to express $\nabla\psi$ in terms of χ and $\partial\beta/\partial t$ and $\partial J/\partial t$:

$$-c\nabla_h\psi = c\frac{\mathbf{R}\cdot\mathbf{B}}{B^2}\mathbf{B_h} - \frac{B_z^2}{W^2B^2}\hat{\mathbf{z}}\times\nabla_h\chi - \frac{1}{W^2B^2}\left(\mathbf{B_h}\cdot\nabla_\mathbf{h}\chi\right)\hat{\mathbf{z}}\times\mathbf{B_h} + \nabla_h\times\dot{\beta}\,\hat{\mathbf{z}} \quad (35)$$

$$-c\frac{\partial\psi}{\partial z} = cB_z\frac{\mathbf{R}\cdot\mathbf{B}}{B^2} - B_z\frac{\nabla_h\chi\cdot\hat{\mathbf{z}}\times\mathbf{B_h}}{W^2B^2} + \dot{J} \quad (36)$$

Summary of current situation

- It is possible to derive a 3D electric field from a time sequence of vector magnetograms which obeys the Maxwell Faraday equation. This formalism uses the Poloidal-Toroidal Decomposition (PTD) formalism.

- The PTD solution for the electric field is not unique, and can differ from the true solution by the gradient of a potential function. The contribution from the potential function can be quite important.

- An equation for a potential function, or for a combination of the PTD electric field and a potential function, can be derived from iterative techniques and from variational techniques.

- The development and testing of numerical techniques for solving the variational techniques are currently underway.