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Hydromagnetic discontinuities from the evolution of nonlinear Alvén waves

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Abstract. To study the large-amplitude Alfvén waves that are frequently observed in the solar wind, the nonlinear equations of electron and ion fluids coupled to Maxwell equations have been solved to obtain the relationships between the fluid density and magnetic field for finite amplitude Alfvén waves in the cold plasma limit. Two regimes of the parameter ω_c/Ω (ratio of characteristic frequency to the ion Larmor frequency) are examined. When $\omega_c/\Omega = 0$, we obtain an exact simple wave solution for the linearly polarized Alfvén wave, and recover the earlier results of [*Montgomery*, 1959a]. For finite ω_c/Ω, the geometric optics approximation which is valid for large wave numbers yields a new relationship which involves elliptically polarized Alfvén waves. Both of these waves can be shown to evolve into shocks.

Introduction

Large-amplitude Alfvén waves are frequently observed in the solar wind [*Belcher and Davis*, 1971] and in the interplanetary discontinuities in regions of high speed streams [*Tsurutani et al.*, 1994]. The nonlinear behavior of collisionless plasmas formulated in a self-consistent way was first studied by *Montgomery*, [1959a], using the Lorentz equation of motion for electrons and ions coupled to Maxwell equations. He used a magnetic field $\mathbf{B} = B_0\mathbf{e}_x + B_y(x,t)\mathbf{e}_y$ and by allowing the plasma density to be a function of space and time, obtained an equation that relates the density and magnetic field (a result obtained using a perturbation technique valid to third order in B_y/B_x). *Montgomery* [1959a] showed that linearly polarized Alfvén waves become nonlinear in the course of time and the waves steepen to form shocks. Montgomery's small-amplitude wave analysis was extended to a finite-amplitude limit by *Cohen and Kulsrud* [1974] who used a multiple time scale analysis of ideal magnetohydrodynamic (MHD) equations.

In this paper, a magnetic field $\mathbf{B} = B_0\mathbf{e}_x + B_y(x,t)\mathbf{e}_y + B_z(x,t)\mathbf{e}_z$ is used in the two-fluid equations coupled to Maxwell equations to obtain an exact simple wave solution that relates the density, velocity and magnetic field for linearly polarized Alfvén waves in cold plasmas (the simple wave method has also been used recently to solve nonlinear MHD equations by *Mann*, [1995]). In the limit of "slow" processes, the earlier result of *Montgomery* [1959a] is recovered. However, for an arbitrary fluid time scale, geometric optics approximation yields a new relationship which involves elliptically polarized nonlinear Alfvén waves. These nonlinear waves can also steepen and evolve into shocks. The nonlinear Alfvén waves we investigate behave differently from the large-amplitude circularly polarized waves [*Ferraro*, 1955] and the Alfvén waves that permeate in steady-state problems [*Montgomery*, 1959b; *Kellogg*, 1964; *Kakutani*, 1966; *Lee and Parks*, 1992].

This article presents only the analytical theory and we defer to future works for application to specific space problems. Our formulation does not include thermal effects or wave dissipation mechanisms, and thus no information on heating can be obtained.

Basic Equations and Formulation

We start from two-fluid equations for a collisionless cold plasma and consider processes when the electron inertia can be neglected. Then the equation of motion for electrons reduces to $\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B} = \mathbf{0}$ where $\mathbf{v}$ is the velocity field of the electron fluid. $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields, and assume that plasma is quasi-neutral. Then Ampere's equation becomes $\nabla\times\mathbf{B} = \frac{4\pi e}{c}N(\mathbf{V}-\mathbf{v})$ where $\mathbf{V}$ is the velocity of the ion fluid, N is the density of ions and electrons, and e is the charge of an electron. These equations combined yield

$$\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B} = \frac{1}{4\pi Ne}(\nabla\times\mathbf{B})\times\mathbf{B} \qquad (1)$$

The equation of motion for the cold ions can be written as

$$M(\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}) = \frac{1}{4\pi N}(\nabla\times\mathbf{B})\times\mathbf{B} \qquad (2)$$

where M is the mass of the ion. Taking the curl of Eq. 1 and making use of Faraday's law, $c\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, we obtain

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$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{B} + \mathbf{B}(\nabla\cdot\mathbf{V}) = (\mathbf{B}\cdot\nabla)\mathbf{V}$$
$$-\frac{c}{4\pi e}\nabla\times\{\frac{1}{N}(\nabla\times\mathbf{B})\times\mathbf{B}\} \quad (3)$$

which can be rewritten as

$$(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)\frac{\mathbf{B}}{N} = \frac{\mathbf{B}}{N}\cdot\nabla\mathbf{V}$$
$$-\frac{c}{4\pi N e}\nabla\times\{\frac{1}{N}(\nabla\times\mathbf{B})\times\mathbf{B}\} \quad (4)$$

using the continuity equation for the ions $\frac{\partial N}{\partial t}+\mathbf{V}\cdot\nabla N + N\nabla\cdot\mathbf{V} = 0$. The equation of motion, Eq. 4, and the continuity equation are the basic equations for defining the variables $N(\mathbf{r},t)$,$\mathbf{V}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. We now assume the variables vary only in the x-direction. Then, $\nabla\cdot\mathbf{B} = \partial B_x/\partial x = 0$, and B_x is uniform in space. Thus, assuming that B_x is also constant in time, $\mathbf{B}$ can be represented as $\mathbf{B} = B_0\mathbf{e_x} + B_y(x,t)\mathbf{e_y} + B_z(x,t)\mathbf{e_z} = B_0\mathbf{e_x} + \mathbf{B}_\perp$.

Introduce for convenience the dimensionless variables $t' = \omega_c t$, $x' = \omega_c x/V_A$, $\mathbf{b}_\perp = \mathbf{B}_\perp/B_0$, $v = V_x/V_A$, $\mathbf{v}_\perp = \mathbf{V}_\perp/V_A$ and $n = N/N_0$, where $1/\omega_c$ is an arbitrary characteristic time scale, N_0 is a representative particle density and V_A is the Alfvén wave velocity, $V_A \equiv B_0/\sqrt{4\pi M N_0}$, based on the $x-$component of the ambient magnetic field and N_0, and $\Omega = eB_0/mc$ is the ion gyro-frequency using again the ambient magnetic field. A new set of coordinate variables τ and ξ is then defined by the transformation relation $\frac{\partial}{\partial\tau} \equiv \frac{\partial}{\partial t'} + v\frac{\partial}{\partial x'}$, $\frac{\partial}{\partial\xi} \equiv \frac{1}{n}\frac{\partial}{\partial x'}$. In terms of the new variables the basic equations for a two fluid cold plasma are

$$\frac{\partial n}{\partial\tau} = -n^2\frac{\partial v}{\partial\xi} \quad (5)$$

$$\frac{\partial v}{\partial\tau} = -\frac{1}{2}\frac{\partial}{\partial\xi}(b_y^2 + b_z^2) \quad (6)$$

$$\frac{\partial\mathbf{v}_\perp}{\partial\tau} = \frac{\partial\mathbf{b}_\perp}{\partial\xi} \quad (7)$$

$$\frac{\partial}{\partial\tau}(\frac{\mathbf{b}_\perp}{n}) = \frac{\partial\mathbf{v}_\perp}{\partial\xi} - \frac{\omega_c}{\Omega}\frac{\partial^2}{\partial\xi^2}(\mathbf{e_x}\times\mathbf{b}_\perp) \quad (8)$$

The equations in the coordinate system (ξ,τ) correspond to the Lagrangian description of fluid motion [*Landau and Lifshitz,* 1959].

Analysis

If the density is uniform ($n = 1$, $N = N_0$), Eqs. 7 and 8 constitute a closed set. For very slow processes where $\omega_c/\Omega \to 0$, b_y and b_z are independent of each other. Seeking a solution of the type $b_{y,z} = Re\{\widetilde{b}_{y,z}e^{i(k\xi-\omega\tau)}\}$ yields a dispersion relation $\omega^2 - k^2 = 0$ which is the ordinary Alfvén wave solution. This equation is valid for cold MHD plasma. Cold MHD equations yield the pure Alfvén mode which is not accompanied by a density perturbation and the magnetosonic mode which is connected with density disturbances at oblique propagation angle relative to B_0. We can make one of the components, say b_z, zero by rotating the coordinate system about the x-axis, and express the disturbances in the field as a superposition of waves having a fixed polarization along the y-axis. This is not possible if ω_c/Ω is finite, since then the two equations are coupled. For finite ω_c/Ω, the dispersion relation obtained from Eq. 8 is

$$\omega^2 - k^2 \pm \frac{\omega_c}{\Omega}k^2\omega = 0 \quad (9)$$

where the $\pm$ signs correspond to the eigenmodes $b_\pm \equiv b_y \pm ib_z$. These waves are elliptically polarized Alfvén waves [*Kennel et al.,* 1988].

Now consider the general case of non-uniform density. Rewriting the set of equations (5-8) in terms of $b_\pm$, obtain

$$\frac{\partial n}{\partial\tau} = -n^2\frac{\partial v}{\partial\xi} \quad (10)$$

$$\frac{\partial v}{\partial\tau} = -\frac{1}{2}\frac{\partial}{\partial\xi}(b_+b_-) \quad (11)$$

$$\frac{\partial^2}{\partial\tau^2}(\frac{b_\pm}{n}) = \frac{\partial^2 b_\pm}{\partial\xi^2} \mp i\frac{\omega_c}{\Omega}\frac{\partial^2}{\partial\xi^2}\frac{\partial b_\pm}{\partial\tau} \quad (12)$$

We seek a special class of simple wave solutions. Simple wave solutions retain the nonlinear character of the system of equations. Simple wave solutions to ideal MHD equations have been recently obtained by Mann, [1995].

Simple Waves

Assume that $\mathbf{v}$ and $\mathbf{b}$ are a function of n only. Then Eqs. 10 and 11 become $\frac{\partial n}{\partial\tau} = -n^2\frac{\partial n}{\partial\xi}\frac{dv}{dn}$ and $\frac{\partial n}{\partial\tau}\frac{dv}{dn} = -c^2\frac{\partial n}{\partial\xi}$, where $c^2 \equiv \frac{1}{2}\frac{d}{dn}(b_+b_-)$ which is to be determined after we find b_+b_- as a function of n. These equations yield $n^2(\frac{dv}{dn})^2 = c^2$. Thus, once we know c in terms of n, v can be obtained. Noting that $dn = \frac{\partial n}{\partial\tau}d\tau + \frac{\partial n}{\partial\xi}d\xi$, we obtain $(\frac{\partial\xi}{\partial\tau})_n = -\frac{\partial n}{\partial\tau}/\frac{\partial n}{\partial\xi}$. Also, $\frac{\partial n}{\partial\tau}/\frac{\partial n}{\partial\xi} = -n^2\frac{dv}{dn}$, and we arrive at $(\frac{\partial\xi}{\partial\tau})_n = \pm nc$ which can be integrated immediately to give $\xi = \pm nc(n)\tau + G(n)$ where G is an arbitrary function of n. Denoting the inversion of G by g, we obtain $n(\xi,\tau) = g(\xi \mp nc\tau)$. Since $n(\xi,0) = g(\xi)$, $g(\xi)$ is determined by the initial profile $n(\xi,0)$.

It is instructive to translate what we have obtained into the original space and time coordinates (x',t'). Since $dn = \frac{\partial n}{\partial\tau}d\tau + \frac{\partial n}{\partial\xi}d\xi = (\frac{\partial n}{\partial t'} + v\frac{\partial n}{\partial x'})d\tau + \frac{1}{n}\frac{\partial n}{\partial x'}d\xi$ we obtain $(\frac{\partial\xi}{\partial\tau})_n = -n(\frac{\partial n}{\partial t'} + v\frac{\partial n}{\partial x'})/\frac{\partial n}{\partial x'}$. Further, noting that $(\frac{\partial x'}{\partial t'})_n = -\frac{\partial n}{\partial t'}/\frac{\partial n}{\partial x'}$ we obtain $(\frac{\partial x'}{\partial t'})_n = v + \frac{1}{n}(\frac{\partial\xi}{\partial\tau})_n = v \pm c$. Integration of this equation yields $x' = (v \pm c)t' + F(n)$ where $F(n)$ is an arbitrary function of n. Thus

$$n(x',t') = f(x' - (v \pm c)t') \quad (13)$$

where f is the inverse of F which is also associated with the initial density profile $n(x',0)$. This equation represents a traveling wave propagating with a speed $v \pm c$. Since v is the velocity of the flow, $\pm c$ represents the wave velocity relative to the flow. If $v \pm c$ is an increasing function of n, waves having the form of the equation above steepen when they propagate into a region of lesser density: wave fronts travelling behind have larger speed than the ones ahead and they eventually overtake them and the density gradient becomes steeper as the compressive wave propagates until a discontinuity is formed [*Landau and Lifshitz*, 1959, *Landau et al.*, 1984]. The density gradient $(\partial n/\partial x')_t$ becomes infinite at the point of discontinuity and the shock formation time t_s is given by

$$1/t_s = -\frac{\partial n(x',0)}{\partial x'}\frac{d}{dn}(v \pm c) \tag{14}$$

Simple waves are stationary solutions in the instantaneous wave frame. It is instantaneous because v and c are not uniform, so no single Galilean transformation can bring us to the wave frame. Thus the problem of finding a simple wave solution is equivalent to finding an instantaneous wave frame in which the system of partial differential equations has a stationary solution.

Wave Frame

Introduce an instantaneous Galilean transformation defined by $\frac{\partial}{\partial\zeta} \equiv \frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\sigma} \equiv \frac{\partial}{\partial\tau} + s(\xi,\tau)\frac{\partial}{\partial\xi}$ where $s(\xi,\tau)$ is the relative velocity of the coordinates which is to be determined, and then express the given set of partial differential equations in this new coordinate system. Using the inverse of the transformation $\frac{\partial}{\partial\xi} = \frac{\partial}{\partial\zeta}$, $\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\sigma} - s(\zeta,\sigma)\frac{\partial}{\partial\zeta}$, Eqs. 10-12 become

$$\frac{\partial n}{\partial\sigma} - s\frac{\partial n}{\partial\zeta} = -n^2\frac{\partial v}{\partial\zeta} \tag{15}$$

$$\frac{\partial v}{\partial\sigma} - s\frac{\partial v}{\partial\zeta} = -\frac{1}{2}\frac{\partial}{\partial\zeta}(b_+b_-) \tag{16}$$

$$\left(\frac{\partial}{\partial\sigma} - s\frac{\partial}{\partial\zeta}\right)\left(\frac{\partial}{\partial\sigma} - s\frac{\partial}{\partial\zeta}\right)\left(\frac{b_\pm}{n}\right) = \frac{\partial^2 b_\pm}{\partial\zeta^2} \mp i\frac{\omega_c}{\Omega}\frac{\partial^2}{\partial\zeta^2}\left(\frac{\partial b_\pm}{\partial\sigma} - s\frac{\partial b_\pm}{\partial\zeta}\right) \tag{17}$$

Now impose the stationarity condition, $\frac{\partial n}{\partial\sigma} = 0$, $\frac{\partial v}{\partial\sigma} = 0$ and $\frac{\partial b_\pm}{\partial\sigma} = 0$, which reduces these equations to

$$s\frac{\partial n}{\partial\zeta} = n^2\frac{\partial v}{\partial\zeta} \tag{18}$$

$$s\frac{\partial v}{\partial\zeta} = \frac{1}{2}\frac{\partial}{\partial\zeta}(b_+b_-) \tag{19}$$

$$s\frac{\partial}{\partial\zeta}\left(s\frac{\partial}{\partial\zeta}\right)\left(\frac{b_\pm}{n}\right) + s\frac{\partial s}{\partial\zeta}\frac{\partial}{\partial\zeta}\left(\frac{b_\pm}{n}\right) = \frac{\partial^2 b_\pm}{\partial\zeta^2} \pm i\frac{\omega_c}{\Omega}\frac{\partial^2}{\partial\zeta^2}\left(s\frac{\partial b_\pm}{\partial\zeta}\right) \tag{20}$$

Note the operators $\frac{\partial}{\partial\sigma}$ and $\frac{\partial}{\partial\zeta}$ do not commute. Rewriting Eq. 20 as

$$\frac{\partial}{\partial\zeta}\left\{s^2\frac{\partial}{\partial\zeta}\left(\frac{b_\pm}{n}\right)\right\} = \frac{\partial^2 b_\pm}{\partial\zeta^2} \pm i\frac{\omega_c}{\Omega}\frac{\partial^2}{\partial\zeta^2}\left(s\frac{\partial b_\pm}{\partial\zeta}\right) \tag{21}$$

and integrating it once, obtain

$$s^2\frac{\partial}{\partial\zeta}\left(\frac{b_\pm}{n}\right) = \frac{\partial b_\pm}{\partial\zeta} \pm i\frac{\omega_c}{\Omega}\frac{\partial}{\partial\zeta}\left(s\frac{\partial b_\pm}{\partial\zeta}\right) \tag{22}$$

Here we assumed the boundary condition $\frac{\partial b_\pm}{\partial\zeta} = \frac{\partial n}{\partial\zeta} = 0$ and $n = 1$ when $\zeta = 0$. Multiplying Eq. 22 with $b_\mp$, adding and subtracting the resulting equations, using $b_\pm = be^{\pm i\phi}$ where b and ϕ are real functions, obtain real and imaginary parts

$$\frac{s^2}{n^2}\left(n\frac{\partial b^2}{\partial\zeta} - 2b^2\frac{\partial n}{\partial\zeta}\right) = \frac{\partial b^2}{\partial\zeta} - 2\frac{\omega_c}{\Omega}\frac{\partial}{\partial\zeta}\left(sb^2\frac{\partial\phi}{\partial\zeta}\right) \tag{23}$$

$$\frac{2s^2}{n}\left(b^2\frac{\partial\phi}{\partial\zeta}\right) = 2b^2\frac{\partial\phi}{\partial\zeta} + \frac{\omega_c}{\Omega}\left[\frac{\partial}{\partial\zeta}\left(s\frac{\partial b^2}{\partial\zeta}\right) - 2s\left\{\left(\frac{\partial b}{\partial\zeta}\right)^2 + b^2\left(\frac{\partial\phi}{\partial\zeta}\right)^2\right\}\right] \tag{24}$$

Assume now that b^2 and v are functions of n, and use Eqs. 18, 19, 23 and 24 to find their functional relationship.

Small ω_c/Ω: First consider the case $\omega_c/\Omega = 0$. Eqs. 18, 19, 23 and 24 then simplify to

$$s = n^2\frac{dv}{dn} \tag{25}$$

$$s\frac{dv}{dn} = \frac{1}{2}\frac{db^2}{dn} \tag{26}$$

$$\frac{s^2}{n^2}\left(n\frac{db^2}{dn} - 2b^2\right) = \frac{db^2}{dn} \tag{27}$$

$$\frac{2s^2}{n}\left(b^2\frac{\partial\phi}{\partial\zeta}\right) = 2b^2\frac{\partial\phi}{\partial\zeta} \tag{28}$$

Eqs. 25 and 26 yield a generally valid relation for a cold plasma $s^2 = \frac{n^2}{2}\frac{\partial b^2}{\partial n}$. Substitute this into Eq. 27 and integrate, and obtain the solution $b^2 = n^2 - 1$ where the integration constant was chosen such that $b = 0$ when $n = 1$. This result, in the original variables with $B_z = 0$, is $(B_0^2 + B_y^2)^{1/2}/N = const$ which is the same relation deduced by [*Montgomery*, 1959a]. Eq. 28 yields $\frac{\partial\phi}{\partial\zeta} = 0$ which indicates that the phase for this solution between b_y and b_z is constant. Thus, these nonlinear Alfvén waves are linearly polarized.

The transformation velocity s of the local Galilean transformation is $s = \pm n^{3/2}$. The flow velocity also can be found by integrating Eq. 25. Note that $(\frac{\partial\xi}{\partial\tau})_\zeta = -\frac{\partial\zeta}{\partial\tau}/\frac{\partial\zeta}{\partial\xi} = -(\frac{\partial\zeta}{\partial\sigma} - s\frac{\partial\zeta}{\partial\zeta}) = s$ for the case $\frac{\partial n}{\partial\sigma} = 0$. Since $(\frac{\partial\xi}{\partial\tau})_\zeta = (\frac{\partial\xi}{\partial\tau})_n$, we immediately obtain a simple wave solution $n(\xi,\tau) = g(\xi - s\tau)$ or expressed in the coordinate system (x',t') $n(x',t') = f(x' - (v + \frac{s}{n}t'))$. Note that $\frac{s}{n}$ represents the wave velocity relative to the flow in (x',t') space, which was denoted by c previously.

Now consider when $\epsilon \equiv \frac{\omega_c}{\Omega}$ is a small parameter. Since when $\epsilon = 0$, $\frac{\partial\phi}{\partial\zeta} = 0$ assume that $\frac{\partial\phi}{\partial\zeta}$ is also of order ϵ. Then to order in ϵ, Eqs. 23 and 24 reduce to

$$b^2\frac{\partial\phi}{\partial\zeta} = \frac{\epsilon}{2}[\frac{\partial}{\partial\zeta}(s\frac{\partial b^2}{\partial\zeta}) - 2s(\frac{\partial b}{\partial\zeta})^2]/(\frac{s^2}{n} - 1) \quad (29)$$

$$\frac{s^2}{n^2}(n\frac{\partial b^2}{\partial\zeta} - 2b^2\frac{\partial n}{\partial\zeta}) = \frac{\partial b^2}{\partial\zeta} + O(\epsilon^2) \quad (30)$$

Eq. 30 is the same as Eq. 27 to first order in ϵ. Thus we conclude that for a process whose characteristic time scale is large so that $\frac{\omega_c}{\Omega}$ is small, the relation $b^2 = n^2 - 1$ is valid to first order in ϵ.

Finite ω_c/Ω: Since when $\omega_c/\Omega = 0$, $\frac{\partial\phi}{\partial\zeta} = 0$, we can expect that the magnitude of $\frac{\partial\phi}{\partial\zeta}$ increases as $\frac{\omega_c}{\Omega}$ increases. Consider a regime where the variation of the envelope of the wave can be neglected compared to that of the phase. This approximation, called the geometric optics approximation, is valid for large wave numbers. Then, $b^2(\frac{\partial\phi}{\partial\zeta})^2$ in Eq. 24 dominates and it reduces to

$$(\frac{s^2}{n} - 1)b^2 = -\frac{\omega_c}{\Omega}sb^2(\frac{\partial\phi}{\partial\zeta}) \quad (31)$$

Substitution of this relation into Eq. 23 yields

$$\frac{s^2}{n^2}(n\frac{\partial b^2}{\partial\zeta} - 2b^2\frac{\partial n}{\partial\zeta}) = \frac{\partial b^2}{\partial\zeta} + 2\frac{\partial}{\partial\zeta}[(\frac{s^2}{n} - 1)b^2] \quad (32)$$

which can be rewritten as

$$\frac{s^2}{n^2}(n\frac{db^2}{dn} - 2b^2) = \frac{db^2}{dn} + 2\frac{d}{dn}[(\frac{s^2}{n} - 1)b^2] \quad (33)$$

Using the relation $s^2 = \frac{n^2}{2}\frac{\partial b^2}{\partial n}$, Eq. 33 yields the solution (with $b^2 = 0$ where $n = 1$)

$$b^2 = 3(1 - n^{-2/3}) \quad (34)$$

Conclusion

The relationship between the magnitude of the magnetic field and the density of the medium where the nonlinear Alfvén wave has a simple wave type solution can be expressed as

$$b^2 = \frac{2}{\gamma}(n^\gamma - 1) \quad (35)$$

Here γ is 2 for the linear polarized wave and $-2/3$ for the elliptically polarized wave in the case of large wave numbers. The nonlinear wave velocity c, is

$$c = n^{\frac{\gamma-1}{2}} \quad (36)$$

The flow velocity is

$$v = \frac{2}{\gamma - 1}(n^{\frac{\gamma-1}{2}} - 1) \quad (37)$$

The wave described by the equation $n(x', t') = f(x' - (v + c)t')$ steepens when it propagates into less dense regions and a shock-like structure can be formed in a time given by

$$t_s = \frac{2n^{\frac{3-\gamma}{2}}}{\left|\frac{dn(x',0)}{dx'}\right|(\gamma + 1)} \quad (38)$$

The time t_s is positive for both $\gamma = 2$ and $\gamma = -2/3$. Thus a compressive density perturbation could lead to the formation of shock waves for both classes of processes considered. Future work will emphasize the application of these equations to specific space problems.

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