# On the quasi-linear diffusion in collisionless plasmas (to say nothing about Landau damping)

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(Received 3 April 2012; accepted 2 May 2012; published online 15 June 2012)

General quasi-linear diffusion coefficients for nonrelativistic collisionless plasmas are derived for unstable modes and analytically continued to damped modes. Properties of the resulting diffusion are investigated and discussed. © *2012 American Institute of Physics*.
[http://dx.doi.org/10.1063/1.4729317]

## I. INTRODUCTION

Kinetic linear dispersion in collisionless plasmas for damped waves naturally involves an analytical continuation of the dispersion relation from the unstable region.[1] On the nonlinear level, an evolution of kinetic waves may be described to some extend in a quasi-linear approach. This approximation assumes a superposition of incoherent linear modes which at the second order (in wave amplitude) affect the background averaged particle distribution functions.[2,3] For unstable waves, this approximation gives rise to a quasi-linear diffusion; for particles in the electrostatic case, one gets (for symbol definition see Appendix A)

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \mathcal{D} \frac{\partial f}{\partial v}, \tag{1}$$

where the diffusion coefficient $\mathcal{D}$ depends on the wave spectrum through Landau resonances with the different modes as

$$\propto \frac{\gamma}{(\omega_r - kv)^2 + \gamma^2}, \tag{2}$$

with a resonance broadening due to the finite growth rate. The derivation of Eq. (2) is simple but the question of its validity is difficult; such a diffusion can be an intrinsic property of systems with sufficient overlapping of one-mode resonances/trapped region.[4]

The Landau resonances in the limit $|\gamma| \ll |\omega_r|$ become

$$\propto \pi\delta(\omega_r - kv), \tag{3}$$

and the resonance broadening disappears. In this limit, one gets the expression for the damping/growth rate as $\gamma \propto \partial f/\partial v|_{v=\omega_r/k}$ and this approach makes a self-consistent system with energy and linear momentum conservation.[5,6]

In a general case, for instance, for nonpropagating[7] or reactive[8] instabilities, the resonance broadening must be taken into account. In this case for damped modes relation, Eq. (2) gives negative diffusion coefficients and in the limit $\gamma \to 0-$, one gets a negative value of Eq. (3), $\propto -\pi\delta(\omega_r - kv)$, so that there is a discontinuity of the diffusion coefficients at $\gamma = 0$. To overcome these problems, it is sometimes argued that causality[9] requires to take the absolute value of Eq. (2) as

$$\propto \frac{|\gamma|}{(\omega_r - kv)^2 + \gamma^2}. \tag{4}$$

This form has for $|\gamma| \ll |\omega_r|$ the same limit (3) for the two cases $\gamma < 0$ and $\gamma > 0$ but does not generally conserve the energy (see below). In this paper, we show that the quasi-linear diffusion coefficients need to be analytically continued to negative growth rates from the unstable region similar to the procedure for the linear dispersion relation. We start with the derivation of the quasi-linear diffusion coefficients in the simple electrostatic case and generalize these results to the electromagnetic case.[10] In both cases, we check the conservation properties of the resulting quasi-linear diffusion.

The paper is organized as follows: Section II describes the quasi-linear approximation in the electrostatic case. The electrostatic results are generalized to the general electromagnetic case in Sec. III. Obtained results are discussed in Sec. IV.

## II. ELECTROSTATIC CASE

### A. Unstable case

Let us start with a simple electrostatic case and unstable waves. In this case, the dispersion relation for a mode with a wave vector $k$ and a (complex) frequency $\omega = \omega_r(k) + i\gamma(k)$ where $\gamma > 0$ is given implicitly as

$$D(k,\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{k} \int_{-\infty}^{\infty} \frac{\frac{\partial f_s}{\partial v}}{kv - \omega} \mathrm{d}v = 0. \tag{5}$$

For symbol definition, see Appendix A. Assuming the quasi-linear approximation, a superposition of incoherent linear waves and slowly varying particle velocity distribution functions $f_s$,

$$E_1 = \sum_{\text{modes}} \delta E(k,\omega) \mathrm{e}^{ikx - i\omega t}, \tag{6}$$

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$$f_{s1} = i\frac{q_s}{m_s}\sum_{\text{modes}} \delta E(k,\omega)\frac{e^{ikx-i\omega t}}{kv-\omega}\frac{\partial f_s}{\partial v}, \tag{7}$$

satisfying the linear dispersion relation (5) for $f_s$. The linear terms $E_1$ and $f_{s1}$ are assumed to be real and, consequently, $\omega(-k) = -\overline{\omega(k)}$ and $\delta E(-k,-\bar{\omega}) = \overline{\delta E(k,\omega)}$; note that $D(-k,-\bar{\omega}) = \overline{D(k,\omega)}$.

Taking the second order contribution to the changes of the averaged $f_s = f_s(v_\parallel, v_\perp)$ as

$$\frac{\partial f_s}{\partial t} = -\frac{q_s}{m_s}\left\langle E_1\frac{\partial f_{s1}}{\partial v}\right\rangle, \tag{8}$$

where $\langle\rangle$ denotes space and (fast) time averaging leads to the diffusion equation for the particle distribution functions,

$$\frac{\partial f_s}{\partial t} = \frac{\partial}{\partial v}\mathcal{D}_s\frac{\partial}{\partial v}f_s, \tag{9}$$

with

$$\mathcal{D}_s = \frac{q_s^2}{m_s^2}\sum_{\text{modes}}|\delta E(k,\omega)|^2\Im\frac{1}{kv-\omega}. \tag{10}$$

Using the dispersion relation (5), it is easy to show that the total particle momentum is conserved

$$\frac{\partial}{\partial t}\sum_s n_s m_s\int_{-\infty}^{\infty} v f_s \mathrm{d}v = 0. \tag{11}$$

From Eq. (5) also follows that the total (particle and electric) energy is conserved with energy exchanges between particles and waves,

$$\frac{\partial}{\partial t}\sum_s n_s m_s\int_{-\infty}^{\infty}\frac{v^2}{2}f_s\mathrm{d}v = -\epsilon_0\sum_{\text{modes}}\gamma|\delta E(k,\omega)|^2. \tag{12}$$

Note that in the limit $\gamma\to 0+$ the diffusion coefficient becomes

$$\lim_{\gamma\to 0+}\mathcal{D}_s = \frac{q_s^2}{m_s^2}\sum_{\text{modes}}|\delta E(k,\omega)|^2\pi\delta(\omega-kv), \tag{13}$$

and in this limit, the total particle kinetic energy and the electrostatic wave energy are conserved

$$\lim_{\gamma\to 0}\frac{\partial}{\partial t}\sum_s n_s m_s\int_{-\infty}^{\infty}\frac{v^2}{2}f_s\mathrm{d}v = 0, \tag{14}$$

$$\lim_{\gamma\to 0}\frac{\partial}{\partial t}\sum_{\text{modes}}\frac{\epsilon_0|\delta E(k,\omega)|^2}{2} = 0, \tag{15}$$

and there is no energy exchange between particles and waves.

## B. Stable case

In the stable case $\gamma < 0$, the electrostatic dispersion relation may be written in the following form:

$$D = 1 - \sum_s\frac{\omega_{ps}^2}{k}\int_{-\infty}^{\infty}\left[\frac{1}{kv-\omega} + 2\pi i\mathfrak{d}(kv-\omega)\right]\frac{\partial f_s}{\partial v}\mathrm{d}v = 0, \tag{16}$$

where $\mathfrak{d}$ is the complex extension of the Dirac $\delta$ function (see Appendix B). Repeating the quasi-linear procedure with the same form of $f_{s1}$, Eq. (7) leads to the same quasi-linear diffusion coefficients (10); in this case, however, the diffusion coefficients are negative and the residue terms in the dispersion relation (16) lead to a nonconservation of the linear momentum and energy which makes such a choice unphysical. Similarly taking the absolute value[9]

$$\mathcal{D}_s = \frac{q_s^2}{m_s^2}\sum_{\text{modes}}|\delta E(k,\omega)|^2\left|\Im\frac{1}{kv-\omega}\right|, \tag{17}$$

the quasi-linear diffusion does not conserve the linear momentum and energy. Instead, the linear term $f_{s1}$ has to be chosen as an analytical continuation

$$f_{s1} = i\frac{q_s}{m_s}\sum_{\text{modes}}\delta E(k,\omega)e^{ikx-i\omega t} \times\frac{\partial f_s}{\partial v}\left[\frac{1}{kv-\omega} + 2\pi i\mathfrak{d}(kv-\omega)\right]. \tag{18}$$

Here, $f_{s1}$ is assumed to be real and consequently, $\omega(-k) = -\overline{\omega(k)}$ and $\delta E(-k,-\bar{\omega}) = \overline{\delta E(k,\omega)}$; note that $D(-k,-\bar{\omega}) = \overline{D(k,\omega)}$ as in the unstable case.

Taking the second order contribution (8), one gets the diffusion coefficient,

$$\mathcal{D}_s = \frac{q_s^2}{m_s^2}\sum_{\text{modes}}|\delta E(k,\omega)|^2\Im\left[\frac{1}{kv-\omega} + 2\pi i\mathfrak{d}(kv-\omega)\right], \tag{19}$$

where the new term contains the complex extension of the Dirac delta function which directly corresponds to the residue in the dispersion relation (16). Using the diffusion coefficient (19), one recovers the conservation of the linear momentum (11) and the energy (12).

The diffusion coefficients are positive for $\gamma > 0$ as well as in the limit $\gamma\to 0$. The analytically continued coefficients must therefore be positive at least for weak damping rates. The analytically continued diffusion coefficients are difficult to understand from the physical point of view. For weak damping rates, one can estimate the complex Dirac delta part of the diffusion coefficients using the Taylor expansion of the distribution function as

$$\frac{\partial}{\partial v}\Re\mathfrak{d}(kv-\omega)\frac{\partial f_s}{\partial v} \approx \frac{\partial}{\partial v}\delta(kv-\omega_r)\left(\frac{\partial f_s}{\partial v} - \frac{1}{2}\frac{\gamma^2}{k^2}\frac{\partial^3 f_s}{\partial v^3}\right), \tag{20}$$

and we have to the second order in $\gamma$

$$\frac{\partial f_s}{\partial t} = \frac{\partial}{\partial v}\tilde{\mathcal{D}}_s\frac{\partial f_s}{\partial v} + \frac{\partial}{\partial v}\mathcal{H}_s\frac{\partial^3 f_s}{\partial v^3}, \tag{21}$$

where

$$\tilde{\mathcal{D}}_s = \frac{q_s^2}{m_s^2}\sum_{\text{modes}}|\delta E(k,\omega)|^2\Im\left[\frac{1}{kv-\omega} + 2\pi i\delta(kv-\omega_r)\right], \tag{22}$$

$$\mathcal{H}_s = -\frac{q_s^2}{m_s^2}\sum_{\text{modes}}|\delta E(k,\omega)|^2\frac{\gamma^2}{k^2}\pi\delta(kv-\omega_r). \tag{23}$$

This result indicates that the quasi-linear diffusion for modes with nonnegligible amplitudes and damping rates involves higher order derivatives of the velocity distribution function, i.e., hyper-diffusion in the velocity space.

The presented quasi-linear results are continuous when passing from $\gamma > 0$ to $\gamma < 0$. Taking now the limit $\gamma \to 0-$, one recovers the diffusion coefficient (13). A general formulation for the dispersion relation and the diffusion coefficient can be written defining an analytic continuation of $1/x$ in the distributive sense as

$$\mathfrak{C}\frac{1}{x} = \begin{cases} \frac{1}{x} + 2\pi i\mathfrak{d}(x) & \Im x > 0 \\ \mathcal{P}\frac{1}{x} + \pi i\delta(x) & \Im x = 0\,, \\ \frac{1}{x} & \Im x < 0 \end{cases} \tag{24}$$

where $\mathcal{P}$ denotes principal value (compare with Sokhotski-Plemelj formula). In this way, the linear dispersion relation simply reads

$$1 - \sum_s \frac{\omega_{ps}^2}{k}\int_{-\infty}^{\infty}\mathfrak{C}\frac{1}{kv-\omega}\frac{\partial f_s}{\partial v}\mathrm{d}v = 0, \tag{25}$$

and the diffusion coefficient is given by

$$\mathcal{D}_s = \frac{q_s^2}{m_s^2}\sum_{\text{modes}}|\delta E(k,\omega)|^2\Im\left(\mathfrak{C}\frac{1}{kv-\omega}\right). \tag{26}$$

## III. ELECTROMAGNETIC CASE

The electrostatic results can be generalized to a general electromagnetic case with the dispersion relation $\det\mathbf{D} = 0$ where the dispersion tensor D is

$$\mathbf{D} = (k^2c^2 - \omega^2 + \omega_p^2)\mathbf{1} - \boldsymbol{kk}c^2 + \sum_s \omega_{ps}^2\sum_{n=-\infty}^{\infty} \times \int_{\mathbb{R}^3}\mathfrak{C}\frac{1}{k_\parallel v_\parallel - \omega + n\omega_{cs}}\left(k_\parallel\frac{\partial f_s}{\partial v_\parallel} + n\frac{\omega_{cs}}{v_\perp}\frac{\partial f_s}{\partial v_\perp}\right)\mathbf{T}_{ns}\,\mathrm{d}^3v, \tag{27}$$

and the tensor $\mathbf{T}_{ns}$ may be given as

$$\mathbf{T}_{ns} = \begin{pmatrix} \frac{n^2\omega_{cs}^2}{k_\perp^2}J_n^2 & -\frac{in\omega_{cs}}{k_\perp}v_\perp J_nJ_n' & \frac{n\omega_{cs}}{k_\perp}v_\parallel J_n^2 \\ \frac{in\omega_{cs}}{k_\perp}v_\perp J_nJ_n' & v_\perp^2(J_n')^2 & iv_\parallel v_\perp J_nJ_n' \\ \frac{n\omega_{cs}}{k_\perp}v_\parallel J_n^2 & -iv_\parallel v_\perp J_nJ_n' & v_\parallel^2J_n^2 \end{pmatrix}, \tag{28}$$

in the frame where $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$ and $J_n'$ have $\lambda_s = k_\perp v_\perp/\omega_{cs}$ as the argument.

The quasi-linear approximation assumes a superposition of linear modes

$$\boldsymbol{E}_1 = \sum_{\text{modes}}\delta\boldsymbol{E}(\boldsymbol{k},\omega)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}, \tag{29}$$

$$\boldsymbol{B}_1 = \sum_{\text{modes}}\delta\boldsymbol{B}(\boldsymbol{k},\omega)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}, \tag{30}$$

where $\delta\boldsymbol{B}(\boldsymbol{k},\omega) = \boldsymbol{k}\times\delta\boldsymbol{E}(\boldsymbol{k},\omega)/\omega$ and

$$f_{s1} = \frac{iq_s}{m_s}\sum_{\text{modes}}\sum_{n,l=-\infty}^{\infty}\mathfrak{C}\frac{1}{k_\parallel v_\parallel + l\omega_{cs} - \omega} \times \delta\boldsymbol{E}\cdot\left(\boldsymbol{a}_{\parallel ls}\frac{\partial f_s}{\partial v_\parallel} + \boldsymbol{a}_{\perp ls}\frac{\partial f_s}{\partial v_\perp}\right)J_n\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t+i(l-n)\varphi}, \tag{31}$$

where the two vectors $\boldsymbol{a}_{\parallel ls}$ and $\boldsymbol{a}_{\parallel ls}$ may be given in the frame where $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$ as

$$\boldsymbol{a}_{\parallel ls} = \left(\frac{l\omega_{cs}k_\parallel}{\omega k_\perp}J_l, -iJ_l'\frac{k_\parallel v_\perp}{\omega}, \left[1 - \frac{l\omega_{cs}}{\omega}\right]J_l\right)$$
$$\boldsymbol{a}_{\perp ls} = \left(\left[1 - \frac{k_\parallel v_\parallel}{\omega}\right]\frac{l\omega_{cs}}{k_\perp v_\perp}J_l, -i\left[1 - \frac{k_\parallel v_\parallel}{\omega}\right]J_l', \frac{l\omega_{cs}v_\parallel}{\omega v_\perp}J_l\right).$$

The different modes are assumed to satisfy the linear dispersion relation

$$\det\mathbf{D}(\boldsymbol{k},\omega) = 0 \text{ and } \mathbf{D}(\boldsymbol{k},\omega)\cdot\delta\boldsymbol{E}(\boldsymbol{k},\omega) = 0. \tag{32}$$

The linear terms $\boldsymbol{E}_1$, $\boldsymbol{B}_1$, and $f_{s1}$ are assumed to be real and, consequently, $\omega(-\boldsymbol{k}) = -\overline{\omega(\boldsymbol{k})}$, $\delta\boldsymbol{E}(-\boldsymbol{k},-\bar{\omega}) = \overline{\delta\boldsymbol{E}(\boldsymbol{k},\omega)}$, and $\delta\boldsymbol{B}(-\boldsymbol{k},-\bar{\omega}) = \overline{\delta\boldsymbol{B}(\boldsymbol{k},\omega)}$.

Taking the second order contribution to the changes of the averaged $f_s = f_s(v_\parallel, v_\perp)$ as

$$\frac{\partial f_s}{\partial t} = -\frac{q_s}{m_s}\left\langle(\boldsymbol{E}_1 + \boldsymbol{v}\times\boldsymbol{B}_1)\cdot\frac{\partial f_{s1}}{\partial\boldsymbol{v}}\right\rangle, \tag{33}$$

where $\langle\rangle$ denotes space and (fast) time averaging leads to the diffusion equation for the particle distribution functions

$$\frac{\partial f_s}{\partial t} = \frac{\partial}{\partial v_\parallel}\left(\mathcal{D}_{\parallel\parallel s}\frac{\partial f_s}{\partial v_\parallel} + \mathcal{D}_{\parallel\perp s}\frac{\partial f_s}{\partial v_\perp}\right) + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\left(\mathcal{D}_{\perp\parallel s}\frac{\partial f_s}{\partial v_\parallel} + \mathcal{D}_{\perp\perp s}\frac{\partial f_s}{\partial v_\perp}\right). \tag{34}$$

The diffusion coefficients may be given in the following explicit form:[10,11]

$$\mathcal{D}_{IJs} = \frac{q_s^2}{m_s^2} \sum_{\text{modes}} \Im \sum_{n=-\infty}^{\infty} \overline{(\delta \boldsymbol{E} \cdot \boldsymbol{a}_{Ins})} (\boldsymbol{a}_{Jns} \cdot \delta \boldsymbol{E}) \times \mathfrak{C} \frac{1}{k_\parallel v_\parallel + n\omega_{cs} - \omega}, \tag{35}$$

where $I, J = \parallel, \perp$. For weak damping rates, one gets hyper-diffusion terms analogically to the electrostatic case (21).

Using the dispersion relation (27), it is easy to show that the total parallel momentum and energy are conserved

$$\sum_s \frac{\partial p_{s\parallel}}{\partial t} = -2\epsilon_0 \sum_{\text{modes}} \gamma \Re \frac{k_\parallel |\delta \boldsymbol{E}|^2 - \boldsymbol{k} \cdot \overline{\delta \boldsymbol{E}} \delta E_\parallel}{\omega} = -\frac{\partial p_{em\parallel}}{\partial t}, \tag{36}$$

$$\sum_s \frac{\partial \mathcal{E}_s}{\partial t} = -\sum_{\text{modes}} \gamma \left( \epsilon_0 |\delta \boldsymbol{E}|^2 + \frac{1}{\mu_0} |\delta \boldsymbol{B}|^2 \right) = -\frac{\partial \mathcal{E}_{em}}{\partial t}. \tag{37}$$

In the limit $\gamma \to 0$, there is again no exchange of parallel linear momentum and energy between particles and waves as in the electrostatic case.

## IV. DISCUSSION

In this paper, we investigated quasi-linear properties of the proper kinetic modes in collisionless nonrelativistic plasma; the evolution of transient modes[12] due to the initial condition is beyond the scope of the paper. We show that the general quasi-linear diffusion coefficients (with a resonance broadening due to the finite growth rate which are easily derived for unstable modes) need to be analytically continued to negative growth rates in a manner similar to the Landau procedure. We derived the general quasi-linear diffusion coefficients starting in the simple electrostatic case and generalized these results to nonrelativistic magnetized plasma. The derived quasi-linear diffusion conserves the energy and the linear momentum and the exchanges of energy and momentum between fields and particles cease as the growth/damping rates approach zero. The analytic continuation of the diffusion coefficients brings the problem of the physical interpretation similar to the case of the original linear result.[1] Our results indicate that the quasi-linear diffusion of damped waves includes some hyper-diffusion in the velocity space (i.e., terms proportional to higher order derivatives of the particle velocity distribution function). The limiting forms of the quasi-linear diffusion coefficients as $\gamma \to 0$ are convenient for theoretical and numerical purposes. However, in this limit, the energy and momentum exchanges between waves and particles approach to zero and, consequently, the particle diffusion is likely reduced. This is consistent with the reduction of the resonance broadening due to the finite damping/growth rate. Conservation laws naturally constrain the particle diffusion and must be taken into account when analysing particle heating and acceleration by electromagnetic fluctuations.

## ACKNOWLEDGMENTS

This work was supported by the Grant No. P209/12/2032 of the Czech Science Foundation and by the Project Nos. RVO:67985815 and RVO:68378289.

## APPENDIX A: GLOSSARY

Here, $t$ denotes the time and the subscript s denotes different species. Here, $i$ denotes the imaginary unit, $\Re$ and $\Im$ denote the real and imaginary part, respectively, and the overline denotes the complex conjugate, $\overline{a + ib} = a - ib$ for real $a$ and $b$. Here, $\omega$ denotes the complex frequency, $\omega_r = \Re\omega$, $\gamma = \Im\omega$.

In the electrostatic case, $x$ denotes the only spatial variable, $f_s$ denotes the ambient (averaged) normalized velocity distribution function, $f_s = f_s(v)$ where $v$ is the only velocity component; $E_1$ denotes the first-order, linear electric field fluctuations, $\delta E$ denote the linear fluctuations of a given mode, and $f_{s1}$ denotes the first-order, linear fluctuations of the velocity distribution function. Here, $k$ denotes the only component of the wave vector.

In the electromagnetic case, $\boldsymbol{x}$ denotes the position, $\boldsymbol{E}$ and $\boldsymbol{B}$ denote the electric and magnetic fields, respectively, $\boldsymbol{B}_0$ denotes the ambient magnetic field, and $B_0 = |\boldsymbol{B}_0|$ denotes its magnitude. Here, $f_s = f_s(v_\parallel, v_\perp)$ denotes the ambient (averaged) normalized velocity distribution function where $v_\parallel$ and $v_\perp$ denote magnitude of the velocity components parallel and perpendicular to $\boldsymbol{B}_0$, respectively. Here, $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$ denote the first-order, linear fluctuations of the electric and magnetic fields, respectively, and $\delta\boldsymbol{E}$ and $\delta\boldsymbol{B}$ denote the linear electric and magnetic fields of a given linear mode. Here, $f_{s1} = f_{s1}(v_\parallel, v_\perp, \varphi)$, the first-order, linear fluctuations of the velocity distribution function where $\varphi$ denotes the gyrophase. Here, $\boldsymbol{k}$ denotes the wave vector, and $k_\parallel$ and $k_\perp$ denote their parallel and perpendicular components with respect to $\boldsymbol{B}_0$, respectively.

Here, $\omega_{cs} = q_s B_0/m_s$ and $\omega_{ps} = (n_s q_s^2/m_s \epsilon_0)^{1/2}$ denote the cyclotron and plasma frequencies, respectively. In these expressions, $m_s$, $q_s$, and $n_s$ denote the mass, the charge, and the number density, respectively, and $\epsilon_0$ and $\mu_0$ denote the vacuum electric permittivity and magnetic permeability, respectively.

In the electromagnetic case, $p_{s\parallel}$ stands for the particle parallel linear momentum $p_{s\parallel} = m_s n_s \int_{\mathbb{R}^3} v_\parallel f_s \, d^3 v$ and $\mathcal{E}_s$ denotes the particle kinetic energy $\mathcal{E}_s = m_s n_s \int_{\mathbb{R}^3} (v^2/2) f_s \, d^3 v$ while $p_{em\parallel}$ stands for the (averaged) linear momentum of the electromagnetic field $p_{em\parallel} = \epsilon_0 \langle \boldsymbol{E} \times \boldsymbol{B} \rangle \cdot \boldsymbol{B}_0 / B_0$, and $\mathcal{E}_{em}$ denotes the (averaged) electromagnetic energy $\mathcal{E}_{em} = \langle \epsilon_0 |\boldsymbol{E}|^2 + |\boldsymbol{B}|^2/\mu_0 \rangle / 2$ where $\langle\rangle$ denote averaging.

## APPENDIX B: COMPLEX DIRAC $\delta$ FUNCTION

Complex extension of the Dirac $\delta$ function for a sufficiently smooth (analytic) real function $f$ is defined as

$$\int_a^b \mathfrak{d}(x - c) f(x) \mathrm{d}x = \begin{cases} f(c) & \text{if } a \le \Re c \le b \\ 0 & \text{else.} \end{cases}, \tag{B1}$$

for real $a, b, x \in \mathbb{R}$, and general complex $c \in \mathbb{C}$. This definition requires that there exists a (unique) analytic continuation of $f$ to $c$.

The complex extension $\mathfrak{d}$ has the following properties for analytical real functions:

$$\mathfrak{d}(\bar{c}) = \overline{\mathfrak{d}(c)}, \tag{B2}$$

which follows from the Taylor expansion around $c_r = \Re c$ in $c_i = \Im c$

$$f(\bar{c}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c_r)\overline{(ic_i)^n}}{n!} = \overline{f(c)}, \tag{B3}$$

where $f^{(n)} = \mathrm{d}^n f/\mathrm{d}x^n$.


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